ELSEVIER

Journal of Fluids and Structures 24 (2008) 1326–1338

JOURNAL OF FLUIDS AND STRUCTURES

www.elsevier.com/locate/jfs

Asymptotic theory of turbulent bluff-body separation: A novel shear layer scaling deduced from an investigation of the unsteady motion

B. Scheichl*, A. Kluwick

Institute of Fluid Mechanics and Heat Transfer, Vienna University of Technology, Resselgasse 3/E322, A-1040 Vienna, Austria

Received 14 January 2008; accepted 15 July 2008
Available online 4 November 2008

Abstract

A rational treatment of time-mean separation of a nominally steady turbulent boundary layer from a smooth surface in the limit $\mathrm{Re} \to \infty$, where Re denotes the globally defined Reynolds number, is presented. As a starting point, it is outlined why the "classical" concept of a small streamwise velocity deficit in the main portion of the oncoming boundary layer does not provide an appropriate basis for constructing an asymptotic theory of separation. Amongst others, the suggestion that the separation points on a two-dimensional blunt body is shifted to the rear stagnation point of the impressed potential bulk flow as $\mathrm{Re} \to \infty$—which is expressed in a previous related study—is found to be incompatible with a self-consistent flow description. In order to achieve such a description, a novel scaling of the flow is introduced, which satisfies the necessary requirements for formulating a self-consistent theory of the separation process that distinctly contrasts former investigations of this problem. As a rather fundamental finding, it is demonstrated how the underlying asymptotic splitting of the time-mean flow can be traced back to a minimum of physical assumptions and, to a remarkably large extent, be derived rigorously from the unsteady equations of motion. Furthermore, first analytical and numerical results displaying some essential properties of the local rotational/irrotational interaction process of the separating shear layer with the external inviscid bulk flow are presented.

Keywords: Asymptotics; Coherent motion; Gross separation; Perturbation methods; Turbulent boundary layers; Turbulent shear layers

1. Introduction

The rational description of break-away separation of a statistically steady and two-dimensional incompressible turbulent boundary layer flow past an impermeable rigid and smooth surface in the high-Reynolds-number limit represents a long-standing unsolved hydrodynamical problem. Needless to say that an accurate prediction of the position of separation, in combination with the local behaviour of the skin friction, has great relevance for many

*Corresponding author. Tel.: +43 1 58801 32225; fax: +43 1 58801 32299.
E-mail addresses: bernhard.scheichl@tuwien.ac.at (B. Scheichl), alfred.kluwick@tuwien.ac.at (A. Kluwick).

0889-9746/$ - see front matter
doi:10.1016/j.jfluidstructs.2008.07.001

engineering applications, where e.g. internal flows, like those through diffuser ducts, or flows past airfoils play a crucial role.

1.1. Problem formulation and governing equations

The picture of such flows near separation is sketched in Fig. 1. As a basic assumption, the suitably formed global Reynolds number Re is taken to be asymptotically large:

$$\mathrm{Re} \coloneqq \tilde{U}\tilde{L}/\tilde{\nu} \to \infty, \quad \nu \coloneqq \mathrm{Re}^{-1} \to 0. \tag{1}$$

Herein $\tilde{\nu}$, $\tilde{L}$, and $\tilde{U}$ denote, respectively, the (constant) kinematic viscosity of the fluid, a reference length, typical for the geometry of the portion of the surface under consideration, and a characteristic value of the surface slip velocity impressed by the limiting inviscid stationary and two-dimensional irrotational bulk flow, hereafter formally indicated by $\nu = 0$. All flow quantities are suitably non-dimensionalised with $\tilde{L}$, $\tilde{U}$, and the (uniform) fluid density. Let t, p, $\boldsymbol{x} = (s, n, z)$, and $\boldsymbol{u} = (u, v, w)$ be the time, the fluid pressure, the position, and the velocity vector. Here u, v, and w are the components of $\boldsymbol{u}$ in directions of the natural coordinates s, n, and z, respectively, along, normal to, and projected onto the separating streamline $\mathscr{S}$, given by $n = 0$, of the flow in the limit $\nu = 0$. Furthermore, $u_e(s)$ denotes the surface slip velocity in that limit. The origin $s = n = 0$ is chosen as the location S where $\mathscr{S}$ departs from the surface. Thus, $\mathscr{S}$ coincides with the surface contour for $s \leqslant 0$. Also, note that $\mathscr{S}$ has, in general, a curvature of $\mathcal{O}(1)$ for $|s| = \mathcal{O}(1)$.

In coordinate-free form, the Navier–Stokes equations then are written as

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

$$\mathrm{D}_t \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}, \quad \mathrm{D}_t = \partial_t + \boldsymbol{u} \cdot \nabla, \quad \Delta = \nabla \cdot \nabla, \tag{3}$$

where ∇ is the gradient with respect to $\boldsymbol{x}$. They are subject to the common no-slip condition $\boldsymbol{u} = \boldsymbol{0}$ holding at the surface. As a well-known characteristic, the stationary Reynolds-averaged turbulent flow can be expressed in terms of the time-averaged motion. In the following we employ the conventional Reynolds decomposition of any (in general, tensorial) flow quantity q into its time-mean component $\overline{q}$, see Fig. 1, here regarded as independent of z, and the (in time and space) stochastically fluctuating contribution q',

$$q(\boldsymbol{x}, t, \ldots) = \overline{q}(x, y, \ldots) + q'(\boldsymbol{x}, t, \ldots), \quad \overline{q} \coloneqq \lim_{\Theta \to \infty} \frac{1}{\Theta} \int_{-\Theta/2}^{\Theta/2} q(\boldsymbol{x}, t + \theta, \ldots)\, \mathrm{d}\theta. \tag{4}$$

Herein the dots indicate any further dependences of q apart from on $\boldsymbol{x}$ and t, e.g. on Re. Reynolds-averaging of Eqs. (2) and (3) then yields the well-established Reynolds equations (in the case $\partial_z \equiv 0$ of planar time-mean flow):

$$\nabla \cdot \overline{\boldsymbol{u}} = 0, \tag{5}$$

$$\overline{\mathrm{D}}_t \overline{\boldsymbol{u}} = -\nabla \overline{p} - \nabla \cdot \overline{\boldsymbol{u}'\boldsymbol{u}'} + \nu \Delta \overline{\boldsymbol{u}}, \quad \overline{\mathrm{D}}_t = \overline{\boldsymbol{u}} \cdot \nabla. \tag{6}$$

It is further presumed in the subsequent analysis that all components of the Reynolds stress tensor $-\overline{\boldsymbol{u}'\boldsymbol{u}'}$ are, in general, of asymptotically comparable magnitude (assumption of locally isotropic turbulence). Most important, we disregard any effects due to free-stream turbulence. That is, the turbulent motion originates from the relatively thin fully

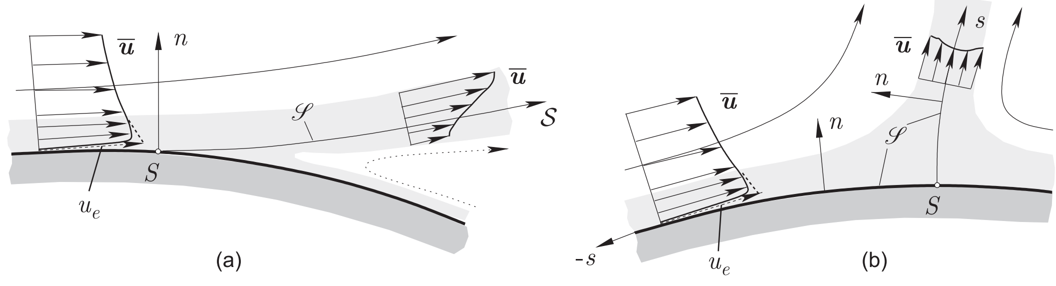

Fig. 1. Time-mean flow near (a) smooth separation (the dotted streamline indicates possible backflow) and (b) separation due to stagnation of the bulk flow, cf. Neish and Smith (1992). The inviscid limit of $\overline{\boldsymbol{u}}$ is shown dashed, and the turbulent shear flow is indicated by a shading.

turbulent boundary layer adjacent to the surface, which near S passes into an accordingly slender separated free shear layer along $\mathscr{S}$ for $s>0$.

1.2. Motivation

From an asymptotic point of view, three outstanding contributions to the solution of the problem under consideration have to be mentioned.

Sychev (1983, 1987) was the first who elucidated the question of the asymptotic structure of the oncoming boundary layer by proposing a three-layer splitting of the latter, sufficiently far ahead of S. This scaling, however, is at variance with the classical finding of a two-tiered boundary layer that is found to hold for firmly attached flow only [see, for instance, the pioneering work by Mellor (1972)]. We start the outline of both formulations by noting that each of them adopts the familiar description of the viscous wall layer close to the surface; the same holds for the flow descriptions discussed subsequently. On top of that region the Reynolds shear stress $-\overline{u'v'}$ asymptotically equals the (local) wall shear stress, given by the square of the skin friction velocity u_*, and the streamwise velocity component $\overline{u}$ satisfies the celebrated logarithmic law of the wall. By using the conventional notation, it reads

$$\overline{u}/u_* \sim \kappa^{-1}\ln n^+ + C^+, \quad n^+ = n\, u_*\,\mathrm{Re} \to \infty, \tag{7}$$

where the well-known constants κ and C^+ are quantities of $\mathscr{O}(1)$. The match of the wall region with the adjacent layer then shows that the expansion

$$[\overline{u}, -\overline{u'v'}/u_*^2] \sim [\overline{u}_0, T_0](s,\eta) - \gamma[U_1, T_1](s,\eta) + \mathscr{O}(\gamma^2), \quad \eta = n/\delta \tag{8}$$

holds in the latter. Here, δ is a measure for the thickness of that layer, and, by introducing the so-called slip velocity u_s, the gauge function γ is seen to satisfy the skin-friction law

$$\gamma = u_*/u_s \sim \kappa/\ln\mathrm{Re}, \quad \mathrm{d}\gamma/\mathrm{d}s = \mathscr{O}(\gamma^2), \quad u_s(s) := \overline{u}_0(s,0). \tag{9}$$

In the classical two-tiered description of the boundary layer, cf. Mellor (1972), it is assumed that in the fully turbulent main region the (positive) streamwise velocity "defect" with respect to the external potential flow, $u_e - u$, is asymptotically small. In turn, $\overline{u}_0(s,\eta) \equiv u_s(s) \equiv u_e(s)$, and in the boundary layer limit the momentum balance (6) reduces to a balance between the linearised convective terms and $\partial_n(-\overline{u'v'})$ in leading order, showing that the boundary layer thickness δ is of $\mathscr{O}(\gamma)$. In contrast, according to the approach made by Sychev (1983, 1987), the expansion (8) holds in the additionally introduced middle layer which meets the requirement that the velocity defect $u_e - u$ and, consequently, $u_e - u_s$ are quantities of $\mathscr{O}(1)$. Thus, in the boundary layer approximation to Eq. (6) the convective terms balance both $\partial_n(-\overline{u'v'})$ and the imposed (adverse) pressure gradient $-u_e\,\mathrm{d}u_e/\mathrm{d}s$, such that the thickness δ of the middle layer is of $\mathscr{O}(\gamma^2)$. This wake-type flow structure then allows for a significant decrease in the wall shear stress according to (9) when u_s tends to zero as $s \to 0_-$ and, moreover, for the occurrence of flow reversal further downstream by adopting a local turbulent/irrotational interaction strategy (without the need of a specific turbulence closure).

One readily finds that the gradients $\partial_n\overline{u}$ in the viscous wall layer and the adjacent layer, described by the expansion (8), match on the basis of the logarithmic behaviour (7) provided that $\partial_\eta\overline{u}_0 \equiv 0$. Unfortunately, this again gives $\overline{u}_0(s,\eta) \equiv u_s(s) \equiv u_e(s)$ and, thus, contradicts the original assumption of a large velocity defect in the middle layer. That inherent mismatch of the wall layer and the wake region was first noted by Melnik (1989), who used mixing length arguments, in the second work to be highlighted. Therefore, Sychev's approach can hardly be accepted as a self-consistent theory. Let us also note the closely related inconsistency encountered in connection with the two-tiered boundary layer proposed by Afzal (1996), who also suggested a velocity deficit of $\mathscr{O}(1)$ to hold in the outer region. However, Melnik also proposed a non-classical initially three-tiered boundary layer where the outermost part plays the role of the aforementioned middle layer. But most important, and in striking difference to any previous asymptotic treatment of turbulent shear flows, in Melnik's (1989) work the slenderness of the latter is measured by some small non-dimensional parameter, denoted by α, which is regarded to be essentially independent of Re. Melnik's motivation for the resultant two-parameter matched asymptotic expansions of the flow quantities merely relies upon the observation that any commonly employed shear stress closure includes a small number (a most familiar example is the so-called Clauser "constant" $\alpha \approx 0.0168$ in the algebraic Cebeci–Smith model) which is seen to measure the boundary layer thickness if the velocity defect in the fully turbulent flow regime is taken to be of $\mathscr{O}(1)$. This idea has been followed up and substantiated by order-of-magnitude reasoning in the more recent papers by Scheichl and Kluwick (2007a, b), where it is shown to provide a sound basis for developing a self-consistent theory of turbulent marginal separation. On the other hand, it is found that Melnik's theory cannot be extended in order to describe the global separation process due to two serious shortcomings: (i) the proposed flow structure is strongly associated with the adopted coupling

$\alpha^{1/2} \ln \mathrm{Re} = \mathcal{O}(1)$, which is apparently inconsistent with the original assumption on α and, hence, does not allow for a correct formulation of the gradual transition from attachment to separation of the flow inside the wall layer; (ii) the impressed potential flow does not exhibit a free streamline departing smoothly from the surface, in order to avoid a Goldstein-type singularity encountered by the boundary layer solution that is evidently unsurmountable by assuming a firmly attached external bulk flow, cf. Scheichl and Kluwick (2007b).

A different viewpoint was taken up in the third contribution to be noticed, by Neish and Smith (1992). They considered the streamwise development of a classical small-defect boundary layer where the irrotational external flow is indeed presumed to be strictly attached; that is, it exhibits a rear stagnation point, see Fig. 1(b). Interestingly, this concept is fully consistent with the following important finding elucidated in the subsequent analysis: in the case of smooth inviscid flow detachment, as depicted in Fig. 1(a), the associated singular behaviour of the surface pressure immediately upstream of the (a priori unknown) position of S does not trigger a significant change in the order of magnitude of the (initially small) velocity defect, which would be necessary to render smooth boundary layer separation possible. Consequently, within the framework of classical turbulent boundary layer theory separation is suggested to occur asymptotically close to the rear stagnation point as $\mathrm{Re} \to \infty$.

Unfortunately, however, it has not been addressed satisfactorily by Neish and Smith (1992) whether and how the small velocity defect may rather abruptly become of $\mathcal{O}(1)$ due to the retardation of the potential flow as the stagnation point S is approached, in order to ensure an uniformly valid flow description. As pointed out in the first part of the present study, the inviscid vortex flow induced in the immediate vicinity of the stagnation point S indeed appears to hamper severely the construction of a self-consistent asymptotic theory. This finding represents the starting point for the subsequent analysis, where it is shown how the closure-independent asymptotic formulation of a turbulent boundary layer having a finite thickness of $\mathcal{O}(\alpha)$, $\alpha \ll 1$, as $\mathrm{Re} \to \infty$ and which may undergo marginal separation, see Scheichl and Kluwick (2007b), can be adapted to that of massive separation. Unlike the theories presented by Melnik (1989) and Neish and Smith (1992), here the formal limit $\alpha = \mathrm{Re}^{-1} = 0$ corresponds to the required class of inviscid flows with free streamlines. Furthermore, we demonstrate how the asymptotic scaling of the (oncoming) flow, which in connection with turbulent marginal separation (Scheichl and Kluwick, 2007b) was based on rather heuristic arguments from a time-averaged point of view, can be deduced by means of a multiple-scales analysis of the equations of motion (2) and (3).

We commence the investigation by considering the evolution of the boundary layer immediately upstream of the surface position S, indicating inviscid separation.

2. Limitations of the small-defect approach

The case where the streamwise velocity defect in the fully turbulent main region of the boundary layer is small, say, $u_e - u = \mathcal{O}(\varepsilon)$, $\varepsilon \ll 1$, is considered first. To be more precise, we assume that $\varepsilon = \gamma$, according to Eqs. (8) and (9) (although the more general assumption $\gamma/\varepsilon = \mathcal{O}(1)$, including $o(1)$, would not alter the following analysis substantially). Therefore, the boundary layer thickness δ is of $\mathcal{O}(\gamma)$ and expanded as

$$\delta/\gamma = \Delta_0(x) + \gamma \Delta_1(x) + \cdots . \tag{10}$$

By setting $U_1/u_e = F_0'(s,\eta)$, $\eta = \mathcal{O}(1)$, the leading-order streamwise momentum equation, supplemented with appropriate boundary and matching conditions, then reads

$$u_e[\mathrm{d}(u_e\Delta_0)/\mathrm{d}s]\eta F_0'' - \Delta_0 \partial_s(u_e^2 F_0') = u_e^2 T_0', \tag{11}$$

$$F_0(s,0) = T_0(s,0) - 1 = 0, \quad F_0' \sim -\kappa^{-1} \ln \eta + \mathcal{O}(1), \quad \eta \to 0, \tag{12}$$

$$F_0'(s,1) = F_0''(s,1) = T_0(s,1) = 0. \tag{13}$$

We mention that in this connection primes denote derivatives with respect to η. Also, it will prove convenient to integrate Eq. (11) with respect to η by using Eq. (12), which gives

$$u_e^2[\mathrm{d}(u_e\Delta_0)/\mathrm{d}s]\eta F_0' - \partial_s(u_e^3 \Delta_0 F_0) = u_e^3(T_0 - 1). \tag{14}$$

Finally, evaluation of relationship (14) at the boundary layer edge and subsequent integration from some value $s_0 < 0$ to $s < 0$ yields

$$\mathrm{d}[u_e^3 \Delta_0 F_0'(s,1)]/\mathrm{d}s = u_e^3, \quad u_e^3(\sigma)\Delta_0(\sigma)F_0'(\sigma,1)|_{\sigma=s_0}^{\sigma=s} = \int_{s_0}^{s} u_e^3(\sigma)\,\mathrm{d}\sigma. \tag{15}$$

In order to assess the assumption of a small velocity defect holding in the oncoming flow with respect separation, we analyse Eqs. (11)–(13) in the limit $s \to 0_-$ for the two different cases indicated by Fig. 1(a) and (b), respectively. Without adoption of a specific turbulence closure, we begin the analysis by considering the first case.

2.1. Flow slightly upstream of smooth separation

It is well known that, under rather general conditions concerning the flow in the stagnant (i.e.dead-water) or backflow region where $s>0$ and $n<0$,

$$u_e(s)/u_e(0) \sim 1 + 2k(-s)^{1/2} + (10k^2/3)(-s) + \mathcal{O}\left[(-s)^{3/2}\right], \quad s \to 0_-, \tag{16}$$

in the inviscid limit $\nu = 0$, cf. Imai (1953), Birkhoff and Zarantonello (1957), and Gurevich (1966), for instance. Here the non-negative parameter k parametrises the class of smoothly separating flows as it depends on the position of S on the body contour. It gives rise to a locally adverse and unbounded pressure gradient $-u_e\,\mathrm{d}u_e/\mathrm{d}s \sim k(-s)^{-1/2}$. Therefore, the question arises if the latter provokes a significant increase of the velocity defect in the oncoming boundary layer, which is required for a correct description of flow reversal further downstream. We remark that in the references mentioned above the free streamline is assumed to confine a dead-water zone (Kirchhoff-type potential flow) throughout. However, it can be demonstrated that Eq. (16) holds in the more general case of a (smoothly) separating potential flow that exhibits a relatively weak backflow. We disregard this possibility in the following, since we feel that for a physically realistic flow picture a reverse flow eddy is necessarily associated with viscous effects.

In order to keep the analysis as general as possible, we now only assume that

$$u_e(s)/u_e(0) \sim 1 + \chi(s) + \cdots, \quad |\mathrm{d}\chi/\mathrm{d}s| \to \infty, \quad s \to 0_-. \tag{17}$$

This singular behaviour is expected to provoke a considerable growth of the turbulent velocity scale u_* (and, in turn, of the fluctuations), expressed through a gauge function $\varphi(s)$,

$$F_0 \sim \varphi(s)G(\eta) + \cdots, \quad T_0 \sim \varphi^2(s)R(\eta) + \cdots, \quad \varphi \to \infty, \quad s \to 0_-. \tag{18}$$

From Eqs. (15) and (18), and the fact that u_e in (17) admits a finite limit, there follows a (intuitively rather unexpected) decrease of the boundary layer thickness of the form $\Delta_0 \sim D/\varphi$, where D is a (positive) constant. Also note that the term $\partial_s(u_e^3\Delta_0F_0)$ in Eq. (14) is bounded for $s \to 0_-$. Since u_e is bounded, too, the first term in Eq. (14) asymptotically equals $-Du_e^3(0)\eta G'(\eta)\,\mathrm{d}(\ln\varphi)/\mathrm{d}s$. As the velocity defect and, in turn, G' are non-negative, that expression tends to $-\infty$ for $s \to 0_-$. Then φ is seen to be proportional to $(-s)^{-1/2}$, as relationship (14) reduces to a balance between that negative term and $u_e^3\,\varphi^2(s)R(\eta)$. The latter term, however, is non-negative, as is the Reynolds stress T_0 in the oncoming flow. From this contradiction one then infers that F_0, T_0, and Δ_0 are finite for $s \to 0_-$. Consequently, inspection of Eqs. (14) and (17), subject to the condition (13), shows that Eq. (18) is to be replaced by a sub-expansion of the expansion (8),

$$[F_0, T_0, \Delta_0] \sim [F_{00}(\eta), T_{00}(\eta), \Delta_{00}] + \chi(s)[F_{01}(\eta), T_{01}(\eta), \Delta_{01}] + \cdots. \tag{19}$$

Therefore, the velocity defect does not change its order of magnitude. One then concludes that, by specifying $\chi(s)$ in Eq. (17) in accordance with the behaviour given by Eq. (16), the small-defect formulation represents an inadequate description of a turbulent boundary layer approaching smooth separation. Note that the same conclusion can be drawn for turbulent separation at a trailing edge under angle of attack, where the external velocity admits a square-root behaviour akin to that in Eq. (16). More generally spoken, the expansion (19) holds if u_e admits a finite limit, according to Eq. (17). We add that it has been demonstrated numerically by Scheichl (2001) that even in case of a rather sharp step-like decrease of $u_e(s)$ the velocity defect characterised by F_0, T_0, and Δ_0, remains bounded.

Summarising, it is possible to give a rather comprehensive answer to an interesting question raised in the comment on the work of Neish and Smith by Degani (1996), namely, how the small-defect structure responds to different limiting forms of $u_e(s)$ as $s \to 0_-$: apparently, the only scenario that is compatible with a change of magnitude of the velocity defect, as it is required for an asymptotic description of separation, is that of a boundary layer approaching a stagnation point of the (otherwise attached) flow in the inviscid limit $\nu = 0$. This is exactly the picture of separation originally proposed by Neish and Smith (1992).

2.2. Flow in the vicinity of a rear stagnation point: non-existence of a self-consistent flow picture

Close to a rear stagnation point, see Fig. 1(b), the potential flow is linearly retarded as $u \sim -cs$, $v \sim cn$, where $s, n \to 0$ and c is a positive constant. Then $u_e \sim -cs$, in contrast to Eq. (17). Substitution of that relationship into the expressions in Eqs. (14) and (15) then predicts a growth of both the boundary layer thickness and the velocity defect, as expressed by Eq. (18). Specifically,

$$\Delta_0 \sim D[-\ln(-s)]^{1/2}/(-s), \quad \varphi = D/\left\{2[-\ln(-s)]^{1/2}(-s)^2\right\}, \; s \to 0_-, \tag{20}$$

where D again is a positive constant, cf. Neish and Smith (1992) and Degani (1996). It then follows from Eq. (20) that relationship (14) reduces to the equation $\eta G'(\eta) = R(\eta)$ for $\eta = \mathcal{O}(1)$. Since the scalings represented by Eq. (20) are incompatible with the inhomogeneous boundary conditions (12) required by the match with the viscous wall layer, on top of the latter a sublayer where $\eta = \mathcal{O}(\varphi^{-2})$ has to be introduced. However, as that flow region appears to behave passively with respect to the further analysis, it is disregarded here.

As a consequence of the growth of Δ, see Eq. (20), the boundary layer approximation ceases to be valid close to the stagnation point S when the distance $-s$ and δ become of comparable magnitude. From the expansion (10) then follows that this region is characterised by suitably rescaled coordinates $(X, Y) = (s, n)/\tau$, where $\tau = (D\gamma)^{1/2}[-(\ln\gamma)/2]^{1/4}$. The resulting asymptotic splitting of the flow is depicted in Fig. 2(a). In the new "square" domain II of extent τ the flow quantities are expanded in the form

$$\left[\frac{\bar{u}}{c\tau}, \frac{\bar{v}}{c\tau}, \frac{\bar{p} - \bar{p}_S}{(c\tau)^2}\right] \sim \left[-X, Y, \frac{X^2 + Y^2}{2}\right] + \frac{1}{\ln\gamma}[\partial_Y \Psi, -\partial_X \Psi, P] + \mathcal{O}\left[\frac{1}{\ln^2\gamma}\right], \tag{21}$$

where $\bar{p}_S$ is the (time-mean) pressure in S. Here the magnitude of the velocity defect is still asymptotically small and varies only logarithmically with γ. As an important implication, the presence of the logarithmic terms in relationships (20) is seen to prevent the Reynolds stresses, which are of $\mathcal{O}(\tau^2/\ln^2\gamma)$, to affect even the perturbed flow in leading order. Indeed, substitution of the expansion (21) into the momentum equation (6) shows that the perturbation stream function $\Psi(X, Y)$ and the pressure disturbance $P(X, Y)$ satisfy the Euler equations, linearised about the stagnant potential flow:

$$\partial_X(X\partial_Y\Psi) - Y\partial_{YY}\Psi = -\partial_X P, \quad -X\,\partial_{XX}\Psi + \partial_Y(Y\,\partial_X\Psi) = -\partial_Y P. \tag{22}$$

By introducing the vorticity $\Omega = (\partial_{XX} + \partial_{YY})\Psi$, elimination of P in Eq. (22) yields the vorticity transport equation, $(X\partial_X - Y\partial_Y)\Omega = 0$. Finally, integration gives

$$(\partial_{XX} + \partial_{YY})\Psi = \Omega(-XY), \tag{23}$$

expressing the well-known property of two-dimensional steady inviscid flows that the vorticity is constant along a streamline. The match with the oncoming boundary layer flow according to Eqs. (18), (20), and (21), and the obvious requirement that the contribution originating from that "square" region to the external potential flow conforms in magnitude to that induced by the incident boundary layer, which is of $\mathcal{O}(\gamma^2)$, then fixes both the vorticity Ω and the boundary conditions supplementing Eq. (23),

$$\Omega = G''(\eta), \quad \eta = -XY, \tag{24}$$

$$\Psi(X, 0) = 0, \tag{25}$$

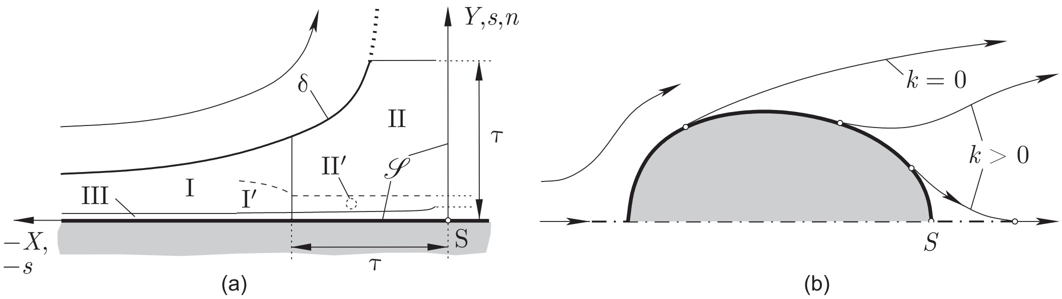

Fig. 2. (a) Asymptotic flow splitting near rear stagnation point S: oncoming boundary layer I with emerging sublayer I′, resulting "square" region II with sublayer II′ (not considered in text), viscous wall layer III (increase of thickness proportional to $1/s$, $s \to 0$, not discussed here), separating streamline $\mathcal{S}$ of the stagnant potential flow; the dotted lines indicate the connection to the regions not considered in the analysis. (b) Smooth inviscid separation from a (here symmetrical) cylindrical body, separating streamlines $\mathcal{S}$ for different values of k in Eq. (16); note the flow showing a cusp-shaped closed cavity which neighbours the attached flow characterised by a rear stagnation point S.

$$\Psi \sim G(\eta)/X^2, \quad \eta = \mathcal{O}(1), \quad X \to -\infty, \tag{26}$$

$$\Psi = \mathcal{O}(r^{-2}) \quad \text{or} \quad o(r^{-2}), \quad r := (X^2 + Y^2)^{1/2} \to \infty. \tag{27}$$

Note that $G'(\eta) \sim G'(0) - (2/\kappa)[G'(0)\eta]^{1/2}$, allowing for a match with the aforementioned sublayer, and not $G'(\eta) \sim -\kappa^{-1} \ln \eta + O(1)$, in accordance with the usual near-wall behaviour given by Eq. (12). One then reveals a finite surface slip velocity given by $U_s(X) := (\partial_Y \Psi)(X, 0)$ and a corresponding half-power behaviour $\partial_Y \Psi \sim U_s - (2/\kappa)[G'(0) Y/X^{1/2}$ as $Y \to 0$. Also, the reuse of the boundary layer coordinate η introduced before in (24) shows that the edge $n = \delta$ of the turbulent flow region II here is given by $\delta \sim \tau/(-X)$, see Fig. 2(a). Stated equivalently, the curve $-XY = \eta \sim 1$ disjoins the turbulent from the (approximately) irrotational external region as $\Omega = 0$ for $\eta \geqslant 1$.

We seek the solution Ψ of the Poisson problem given by Eqs. (23)–(26) in the range $X < 0,\ Y \geqslant 0$. That is, in the present investigation we do not take into account the "collision" of the oncoming flow with that approaching S for $s \to 0_+$, cf. Figs. 1(b) and 2(a). We conveniently set $\Psi = \Psi_p + \Psi_h$, where $\Psi_p(X, Y)$ is a particular solution of Eqs. (23)–(25) and the homogeneous contribution $\Psi_h(X, Y)$ satisfies Laplace's equation, $(\partial_{XX} + \partial_{YY})\Psi_h = 0$, subject to (25). By defining $G(-\eta) := -G(\eta)$, $\eta \geqslant 0$, and using standard methods, one then obtains

$$\Psi_p = \frac{1}{4\pi} \oint_{-\infty}^{\infty} \oint_{-1/|s|}^{1/|s|} G''(-\sigma\upsilon) \ln[(X - \sigma)^2 + (Y - \upsilon)^2] \, d\sigma \, d\upsilon \quad (-\infty < X < \infty, -\infty < Y < \infty), \tag{28}$$

and, after integration by parts and some manipulations

$$\Psi_p = \frac{1}{2\pi} \int_{-1}^{1} G'(\eta) \oint_{-\infty}^{\infty} \frac{|\sigma| Y - \eta}{\sigma[\sigma^2 (X - \sigma)^2 + (|\sigma| Y - \eta)^2]} \, d\sigma \, d\eta \quad (-\infty < X < \infty, -\infty < Y < \infty). \tag{29}$$

The function $\Psi_p(X, Y)$ is skew-symmetric with respect to the origin $X = Y = 0$, where $\Psi_p(0, Y) \equiv \Psi(X, 0) \equiv 0$. Moreover, it is found to vary with R^{-2} for $R^2 = X^2 + Y^2 \to \infty$ and fixed values of $\vartheta := \arctan(Y/X)$ (note that the skew-symmetric distribution of $\Omega(-XY)$ acts like a quadrupole in the far field). On the other hand, $\Psi_p \sim [G(\eta) - H(\eta)]/X^2$ for $X \to -\infty$, with $H \not\equiv 0$ but $H'' \equiv 0$, where η and, in turn, the function $H(\eta)$ (which is not stated explicitly here) are kept fixed. Since $H \not\equiv G$, however $\Psi_h(X, Y)$ must behave as

$$\Psi_h \sim H(\eta)/X^2 (H \not\equiv 0, H'' \equiv 0), \quad \eta = O(1), X \to -\infty, \tag{30}$$

such that Ψ satisfies the upstream condition (26). An asymptotic investigation of Laplace's equation (that takes into account symmetry properties of the solution that are consistent with the aforementioned skew-symmetry of Ψ_p) then shows that $\Psi_h \sim R^{-2}[A \cos(2\vartheta) + B \sin(2\vartheta)]$, where A and B are constants, is the only possible behaviour for $R \to \infty$. Unfortunately, however, and despite its agreement with the far-field conditions (27), this relationship for Ψ_h does not meet the required match with the limiting form (30) as $\vartheta \to \pi_-$ and $X \to -\infty$, Ψ_p is evaluated by exploiting Eq. (29). Thus, the problem posed by Eqs. (23)–(26) has no solution. Therefore, that asymptotic picture of separation taking place close to a rear stagnation point, as originally proposed by Neish and Smith (1992), must be regarded as at least questionable.

A more concise proof of this statement has been found in the course of a private communication with Professor F.T. Smith after submission of the paper (note Acknowledgement): a convenient treatment of the stagnating potential flow considered here is provided by the suitable conformal mapping $\Xi = -Z^2/2$ of the third quarter $(X \leqslant 0,\ Y \geqslant 0)$ of the complex plane $Z := X + \mathrm{i}Y$ onto the upper half of the complex plane $\Xi := \zeta + \mathrm{i}\eta$. In turn, Eqs. (23)–(27) subject to the according transformations $\zeta = (Y^2 - X^2)/2$, $\eta = -XY$, and $\hat{\Psi}(\zeta, \eta) := \Psi(X, Y)$ read

$$(\partial_{\zeta\zeta} + \partial_{\eta\eta})\hat{\Psi} = G''(\eta)/[2(\zeta^2 + \eta^2)^{1/2}], \tag{31}$$

$$\hat{\Psi}(\zeta, 0) = 0, \tag{32}$$

$$\hat{\Psi} \sim -G(\eta)/(2\zeta), \quad \eta = \mathcal{O}(1), \ \zeta \to -\infty, \tag{33}$$

$$\hat{\Psi} = \mathcal{O}(1/\rho) \quad \text{or} \quad o(1/\rho), \quad \rho := (\zeta^2 + \eta^2)^{1/2} \to \infty. \tag{34}$$

In that case a particular solution $\hat{\Psi} = \hat{\Psi}_p(\zeta, \eta)$ of Eqs. (31)–(33) found by exploiting standard methods and the aforementioned anti-symmetry of $\Omega(\eta)$ with respect to $\eta = 0$ is given by

$$\hat{\Psi}_p = \frac{1}{8\pi} \oint_{-1}^{1} G''(\upsilon) \, \mathrm{PV} \int_{-\infty}^{\infty} \frac{\ln[(\xi - \sigma)^2 + (\eta - \upsilon)^2]}{(\sigma^2 + \upsilon^2)^{1/2}} \, d\sigma \, d\upsilon \quad (-\infty < \zeta < \infty, -\infty \leqslant \eta < \infty). \tag{35}$$

Herein, the principal value of the second integral refers to the infinitely remote point, $|\zeta| = \infty$, due to the logarithmic singularity of the integrand there. On the one hand, non-existence of the solution of Eqs. (31)–(34) is indicated by considering the far-field behaviour of an appropriate homogeneous solution $\hat{\Psi} = \hat{\Psi}_h(\zeta, \eta)$ of Eq. (31), in a manner analogous to that adopted in case of the original problem, Eqs. (23)–(27). More simply, however, multiplying of Eq. (31) with η and subsequent integration in the range $-\infty < \zeta < \infty$, $0 \leqslant \eta < \infty$ gives the contradiction $0 \sim (\ln A) \int_0^1 \eta G''(\eta)\,d\eta \equiv -G(1) \ln A$ as $A \to \infty$. Here this limit is considered as integration of the right-hand side of Eq. (31) is initially carried out from $\zeta = -A$ to $\zeta = A$ and $\eta = 0$ to $\eta \to \infty$.

The formal inconsistency outlined before has not been addressed by Neish and Smith (1992). Apparently, this is due to the neglect of the logarithmic terms in expressions (20) in their discussion of the match with the "square" region II. In turn, they propose a vortex flow there which exhibits a velocity defect relative to the stagnating external flow of $\mathcal{O}(1)$, in striking contrast to the expansion (21). Consequently, in the papers by Neish and Smith (1992) and Degani (1996) both the magnitude of the velocities and the extent of the emerging region II are of $\gamma^{1/2}$. Thus, the flow there is governed by the full Reynolds equations (1), rather than their linearised form (22). It is that fully nonlinear stage which prompted Neish and Smith (1992) and Degani (1996) to conclude that separation would occur a distance of $\mathcal{O}(\gamma^{1/2})$ upstream of S. Also, it is not explained in these papers how the flow region II is transformed into a turbulent shear layer along the separated streamline $\mathscr{S}$, which then coincides with the Y-axis, see Fig. 2(a).

A further uncertainty is raised by another issue that has been put forward by Neish and Smith (1992): it is argued that the position of smooth flow detachment approaches the rear stagnation point if one considers the limit $k \to \infty$ in relationship (16). The flow situation for different values of k is sketched in Fig. 2(b), cf. Birkhoff and Zarantonello (1957): from a topological point of view, the only candidate for a flow exhibiting free streamlines around a cylindrical body that neighbours the completely attached potential flow with a rear stagnation point S is the one which embeds a vanishingly small interior (cusp-shaped) cavity/eddy in the vicinity of S. However, it has not been demonstrated convincingly so far that such a solution is associated with correspondingly large values of k. We note that the class of inviscid flows having free streamlines is currently under investigation. [To this end, the methods of potential flow theory presented by Gurevich (1966) are adopted.]

3. The large-defect boundary layer and smooth separation

The picture of separation considered by Neish and Smith (1992) and Degani (1996) is apparently not in accordance with experimental findings. In fact, separation from a cylindrical body takes place a relatively short distance downstream of the location of its maximum cross-section, even for very large values of Re. This finding, together with the serious difficulties discussed in the previous section, then strongly suggests to abandon the assumption of a small-defect boundary layer in favour of a flow description where a streamwise velocity deficit of $\mathcal{O}(1)$ is stipulated. As outlined in the Introduction, such an asymptotic concept that (i) surmounts the difficulties in the matching procedure due to the logarithmic velocity distribution (7) encountered in Sychev's (1983, 1987) theory, and (ii) is corroborated by any commonly used turbulence closure, has already been proven successful in the description of turbulent marginal separation, see Scheichl and Kluwick (2007a, b).

In this novel flow description the boundary layer thickness δ is measured by a small parameter α which is independent of Re as Re $\to \infty$. This most remarkable characteristic anticipates the existence of a turbulent shear layer of finite width with a wake-type flow, even in the formal limit $\alpha \to 0$, $v = 0$, included in the fundamental assumption (1). In that limit the unsteady flow in the wake region is presumably not affected significantly by the periodically occurring well-known wall layer bursts. As will turn out, this characteristic allows for an investigation of some properties associated with the unsteady motion on the basis of Eqs. (2) and (3).

3.1. The slender-wake limit

In the wake region the Reynolds stresses are quantities of $\mathcal{O}(\alpha)$. Then the nonlinearities in the momentum equation (3) suggest to expand the flow quantities according to

$$[u, v, w, p] \sim [\bar{u}_0, 0, 0, \bar{p}_0](s, N) + \alpha^{1/2}[u'_1, v'_1, w'_1, p'_1](t, s, N, \ldots) + \alpha\{[\bar{u}_2, \bar{v}_2, 0, \bar{p}_2](s, N) + [u'_2, v'_2, w'_2, p'_2](t, s, N, \ldots)r\} + \mathcal{O}(\alpha^{3/2}), \quad \delta/\alpha = \delta_0(s) + \mathcal{O}(\alpha^2). \tag{36}$$

Herein a suitable shear layer coordinate $N = n/\alpha$ is introduced, and the dots indicate dependences on inner spatial and time scales, which are specified later. Note that the omission of the time-averaged terms of $\mathcal{O}(\alpha^{1/2})$ in Eq. (36) is a consequence of the expansion $[\bar{u}, \bar{v}, \bar{p}] \sim [\bar{u}_0, \bar{p}_0, 0] + \alpha[\bar{u}_2, \bar{p}_2, \bar{v}_2] + \mathcal{O}(\alpha^2)$ of the time-mean flow quantities, as suggested by the governing

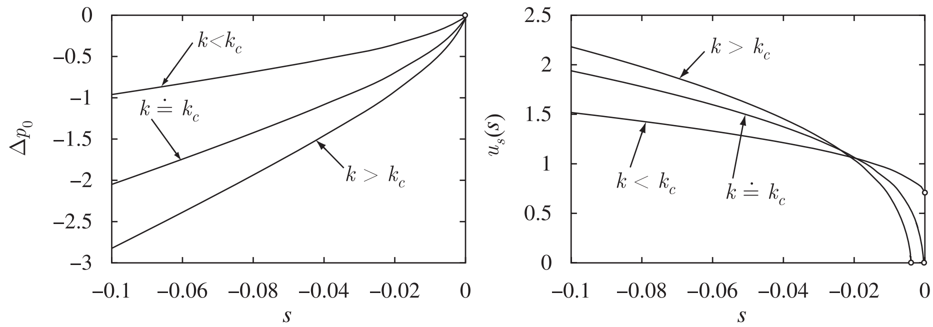

Fig. 3. Local distributions of $\Delta p_0 = p_0(s) - p_0(0)$ and $u_s(s)$ for $k = 1.5$, $k = 2.7 \doteq k_c$, and $k = 3.4$; the circles indicate the occurrence of singular points.

equations (5) and (6). Inserting Eq. (36) into Eqs. (5) and (6) then gives rise to the shear layer approximation

$$\bar{p}_0(s, N) = p_0(s), \quad -\mathrm{d}p_0/\mathrm{d}s = u_e\,\mathrm{d}u_e/\mathrm{d}s, \tag{37}$$

$$\partial_s \bar{u}_0 + \partial_N \bar{v}_2 = 0, \quad \bar{u}_0 \partial_s \bar{u}_0 + \bar{v}_2 \partial_N \bar{u}_0 = -\mathrm{d}p_0/\mathrm{d}s - \partial_N(\overline{u_1' v_1'}). \tag{38}$$

Eqs. (37) and (38) are seen to govern the turbulent motion along the separating streamline $\mathscr{S}$ to leading order sufficiently far from S, i.e. for $|s| = \mathcal{O}(1)$, see Fig. 1(a). They are subject to the wake-type boundary conditions

$$\bar{v}_2(s, 0) = \overline{u_1' v_1'}(s, 0) = 0, \quad \bar{u}_0(s, \delta_0(s)) - u_e(s) = \overline{u_1' v_1'}(s, \delta_0(s)) = 0. \tag{39}$$

By excluding the apparent trivial solution $\bar{u}_0 \equiv u_e(s)$, $\bar{v}_2 \equiv \overline{u_1' v_1'} \equiv 0$, which implies a velocity defect of $o(1)$, we seek non-trivial solutions $\bar{u}_0, \bar{v}_2, \delta_0$ of Eqs. (37)–(39) with respect to separation. To this end, it is useful to consider Eqs. (38) and (37) evaluated for $N = 0$,

$$\mathrm{d}(u_s^2 - u_e^2)/\mathrm{d}s = -2[\partial_N(\overline{u_1' v_1'})](s, 0). \tag{40}$$

Herein $u_s(s) = \bar{u}_0(s, 0)$ again denotes the slip velocity. Note that separation is associated with flow reversal further downstream, which, in turn, requires $u_s(0) = 0$. To gain first insight how the boundary layer behaves as $s \to 0_-$, the problem posed by Eqs. (37)–(39) has been solved numerically, by adopting the same algebraic shear stress closure that was employed successfully for the boundary layer calculations carried out by Scheichl and Kluwick (2007b).

We again discard the possibility that the impressed potential flow exhibits a rear stagnation point S, since inspection of Eq. (40), confirmed by the numerical study, shows that then u_s not necessarily approaches zero in the vicinity of S. Therefore, the picture of a "collision" of two boundary layers is apparently not appropriate for describing turbulent separation. Consequently, separation is seen to be associated with a smoothly separating inviscid flow, according to the situation sketched in Fig. 2(b). As was outlined by Birkhoff and Zarantonello (1957) and Gurevich (1966), only flows having $k \geqslant 0$ are topologically possible. A suitable model for the surface velocity $u_e(s)$ that exhibits the then required local behaviour expressed in Eq. (16) is given by $u_e(s) = (3/2 + s)^m [1 + k(-2s)^{1/2}]/(1 + k)$, $-1/2 \leqslant s < 0$, such that $u_e(-1/2) = 1$. Here the exponent m represents an eigenvalue of the self-preserving solution for a given value of $u_s(-1/2)$, which serves as the initial condition for the downstream integration of Eqs. (37)–(39) cf. Scheichl and Kluwick (2007b). Specifically, the value $u_s(0) = 0.95$ has been imposed, yielding $m \doteq -0.3292$. The distributions for the impressed adverse difference pressure $p_0(s) - p_0(0)$ and the resulting slip velocity $u_s(s)$ are plotted in Fig. 3 for different values of the control parameter k. It is found that for sufficiently small values of k the integration terminates in a singular manner at $s = 0$ where u_s assumes a finite limit, i.e. $u_s(0) > 0$. For increasing values of k this threshold decreases, such that it finally vanishes for a critical value of k, say, $k = k_c$. We note that near $k = k_c$ the numerical calculations are very sensitive to slight variations in the value of k; for the specific choice of $u_e(s)$ adopted here one finds that $k_c \doteq 2.7$. For $k > k_c$, however, the solution admits a Goldstein-type singularity at a position upstream of $s = 0$ which has been discussed in more detail by Scheichl and Kluwick (2007b). Here we add that a thorough analytical study of the numerically observed singular behaviour of the boundary layer solutions, also expressed through Eq. (40), is a task of the current research.

As a first, rather remarkable, result, the location of turbulent break-away separation in the double limit $\alpha = \nu = 0$ is seen to be associated with a positive, presumably single-valued, value k_c of k, which has to be found by means of iterative boundary layer calculations. This strikingly contrasts its laminar counterpart, where the so-called Brillouin–Villat condition fixes the position of inviscid flow detachment by the requirement $k = 0$, see Sychev (1972) and Smith (1977). On the other hand, the experimental findings of Tsahalis and Telionis (1975) strongly support the singular behaviour of the turbulent flow in the boundary layer limit there due to a positive value $k = k^*$ as outlined above. Furthermore, the downstream shift of that point for increasing values of k, sketched in Fig. 2(b), explains why, in general, turbulent separation from a cylindrical body takes place further downstream as it is the case when the flow is still laminar. Moreover, first investigations performed by the authors indicate that in the turbulent case the more precise determination of the location of separation for small but finite values of both α and ν is determined by a locally strong rotational/irrotational interaction mechanism, analogous to that proposed by Sychev (1983, 1987).

3.2. Internal structure "derived" from first principles

As a starting point, we consider the well-known transport equation for the time-averaged specific turbulent kinetic energy $\varkappa = \boldsymbol{u}' \cdot \boldsymbol{u}'/2 = (u'^2 + v'^2 + w'^2)/2$, which results from Reynolds-averaging the inner product of $\boldsymbol{u}'$ with the momentum equation (3) by substituting the continuity equation (2),

$$\overline{\mathrm{D}}_t\overline{\varkappa} + \nabla \cdot \overline{(k + p')\boldsymbol{u}'} - \nu\Delta\overline{\varkappa} + \varepsilon_p = -\overline{\boldsymbol{u}'\boldsymbol{u}'}\colon \nabla\overline{\boldsymbol{u}}, \quad \varepsilon_p := \nu\overline{\nabla\boldsymbol{u}'\colon \nabla\boldsymbol{u}'}. \tag{41}$$

Herein ε_p is commonly referred to as turbulent (pseudo-)dissipation. By taking into account Eq. (36), the least-degenerate shear layer approximation of Eq. (41) in the double limit given by $\alpha \to 0$ and $\nu \to 0$ is found to be

$$\partial_N(\overline{p_1'v_1'}) + \varepsilon_p \sim -\overline{u_1'v_1'}\partial_N\overline{u}_0. \tag{42}$$

We integrate this relationship across the shear layer thickness, i.e. from $N = 0$ to $N = \delta_0$. Then the net contribution of the diffusive term on the left-hand side of Eq. (42) is seen to vanish, whereas the resulting net turbulent "production" on the right-hand side is positive and of $\mathcal{O}(1)$ since both the Reynolds shear stress $-\overline{u_1'v_1'}$ and the shear rate $\overline{u}_{0,N}$ are apparently non-negative. Remarkably, then ε_p is a quantity of $\mathcal{O}(1)$ in the formal limit $\nu = 0$.

The quantity ε_p is obtained by Reynolds-, or equivalently, time-averaging according to Eq. (4), the stochastically varying quadratic form $\nabla\boldsymbol{u}'\colon \nabla\boldsymbol{u}'$. By adopting the rather weak assumption that the averaging process leaves its order of magnitude unchanged, we find, with some reservation (due to the at present apparent lack of experimental evidence), that

$$\nabla\boldsymbol{u}' = \mathcal{O}(\nu^{-1/2}) \tag{43}$$

holds for the predominant fraction of intervals of the time t. As the most simple description of the fluctuating motion, we next assume that the turbulent fluctuations are governed by a single spatial "micro-scale", denoted by λ (together with a correspondingly small time scale) apart from the "macro-scales", represented by a streamwise length of $\mathcal{O}(1)$ and the shear layer thickness of $\mathcal{O}(\alpha)$. It then follows from the estimate (43) in combination with Eq. (36) that appropriate "micro-variables" are given by $(t', \boldsymbol{x}') = (t, \boldsymbol{x})/\lambda$ where $\boldsymbol{x}' = (s', n', z')$ and $\lambda = (\nu\alpha)^{1/2}$. That is, the smallest spatial scales are measured by λ. Interestingly, they are asymptotically larger than the (non-dimensional) celebrated Kolmogorov length scale, which is commonly associated with the dissipative small-scale structure of turbulence and given by $(\nu^3/\varepsilon_p)^{1/4}$. It should be mentioned that this novel "micro-scale" is inherently linked to the asymptotic investigation of the equations of motion (2) and (3) as well as the "first moment" of the latter, given by (41). In contrast, the definition of the Kolmogorov scale merely results from dimensional reasoning and, thus, has no profound rational basis.

Hence, the equations of motion (2) and (3) are expanded in the sequence of "inviscid" linear equations

$$\nabla' \cdot \boldsymbol{u}_i' = 0, \tag{44}$$

$$\mathrm{D}_t'\boldsymbol{u}_i' + \boldsymbol{N}_{i-1}' = -\nabla' p_i', \quad \boldsymbol{N}_0' = \boldsymbol{0}, \quad \mathrm{D}_t' = \partial_{t'} + \overline{u}_0(s, N)\partial_{s'}. \tag{45}$$

Here and in the following $i = 1, 2, \ldots$, $\boldsymbol{u}_i' = (u_i', v_i', w_i')$, and ∇' denotes the gradient with respect to $\boldsymbol{x}'$. The inhomogeneous terms $\boldsymbol{N}_i'$ in Eq. (45) are defined by expanding the nonlinear convective operator in Eq. (3), according to Eq. (36),

$$(\boldsymbol{u} \cdot \nabla' - \overline{u}_0\partial_{s'})\boldsymbol{u}' \sim \alpha^{1/2}\boldsymbol{N}_1' + \alpha\boldsymbol{N}_2' + \cdots. \tag{46}$$

Then the vector $\boldsymbol{N}_i'$ depends on the velocity fluctuations $\boldsymbol{u}_j'$ where $j = 1, 2, \ldots, i$. By eliminating the pressure fluctuations p_i' in Eq. (45), the vorticity fluctuations $\boldsymbol{\omega}_i'$ are seen to satisfy the equations

$$\mathrm{D}_t'\boldsymbol{\omega}_i' = -\nabla' \times \boldsymbol{N}_{i-1}', \quad \boldsymbol{\omega}_i' = \nabla' \times \boldsymbol{u}_i'. \tag{47}$$

Thus, $\mathrm{D}_t'\boldsymbol{\omega}_1' = \boldsymbol{0}$, so that $\boldsymbol{\omega}_1'$ depends on the "micro-variables" $\xi' = s' - \overline{u}_0 t'$, n', and z', but not explicitly on t'. In principle, $\boldsymbol{u}_1'$ then can be calculated from its Helmholtz decomposition, given by the distribution of $\boldsymbol{\omega}_1'$ together with the vanishing divergence as expressed by Eq. (44). Therefore, $\boldsymbol{u}_1'$ and, in turn, $\boldsymbol{N}_1'$ also show no explicit dependence on t', giving $\boldsymbol{\omega}_2' = \boldsymbol{C}' - (\nabla' \times \boldsymbol{N}_1')t'$, where $\boldsymbol{C}'$ is a "constant" of integration. The requirement that the first of the expansions (36) must be uniformly valid with respect to the "micro-time" t' then gives rise to the solvability condition $\nabla' \times \boldsymbol{N}'_1 = \mathbf{0}$. As a result, one recursively finds that $\mathrm{D}'_t \boldsymbol{\omega}'_i = \mathbf{0}$ in general, such that the velocity and pressure fluctuations $\boldsymbol{u}'_i$ and p'_i, respectively, depend on ξ', s', n', and z', but, most important, not explicitly on t', and are determined by

$$\nabla' \times \boldsymbol{N}'_i = \mathbf{0}, \quad \nabla' p'_i = -\boldsymbol{N}'_{i-1}. \tag{48}$$

Eqs. (48) describe a stationary motion with respect to ξ', i.e. in a frame of reference which moves with the time-mean streamwise velocity $\overline{u}_0(s, N)$ along the separating streamline $\mathscr{S}$ of the flow in the formal limit $v = 0$, see Fig. 1(a). Note that they comprise the full nonlinear steady Euler equations, satisfied by $\boldsymbol{u}'_1$ and p'_2.

This wave-type transport of the stochastic fluctuations along with the time-averaged flow found from the "micro-scales" analysis is commonly termed as "coherent motion": this contribution to the overall turbulent motion is usually associated with spatio-temporal regularity, whereas the dynamics of the stochastic fluctuations acting on the smaller scales involves its random part. As a further consequence of these considerations, the process of time-averaging according to Eq. (4) is seen to provide a filtering of the fluctuating motion with respect to ξ' and, in turn, rather not only with respect to the "micro-time" t' but also to the streamwise "micro-variable" s'. The view that the statistically stationary turbulent flow depends on the spatial "macro-variables" s and N only is, therefore, supported by an asymptotic investigation of the Navier–Stokes equations (2) and (3) and, subsequently, usual time-averaging.

The relationships (44)–(47) are valid for $i < I$ where the index I signifies contributions to Eq. (36) of $\mathcal{O}(v^{1/2})$. For $i = I$ it follows from Eqs. (2) and (3) that the dynamics of these contributions are affected by the viscous term on the right-hand side of Eq. (3). Also, the normal gradient $\partial_N \overline{u}_0$ then enters the momentum balance as a consequence of the "macro-scale" α which describes the time-mean shear layer approximation. This in turn suggests the introduction of a further set $(t^\alpha, \boldsymbol{x}^\alpha) = (t, \boldsymbol{x})/\alpha$ of "micro-variables". Let ∇^α denote the gradient with respect to $\boldsymbol{x}^\alpha$ and $\boldsymbol{e}_s$, $\boldsymbol{e}_n$, and $\boldsymbol{e}_z$ the unit vectors in the respective directions indicated by the subscripts. We then find

$$\nabla' \cdot \boldsymbol{u}'_I = -\nabla^\alpha \cdot \boldsymbol{u}'_1, \tag{49}$$

$$\mathrm{D}'_t \boldsymbol{u}'_I + \boldsymbol{N}'_{I-1} + \mathrm{D}^\alpha_t \boldsymbol{u}'_1 + \boldsymbol{e}_s v'_1 \partial_N \overline{u}_0 = -\nabla' p'_I - \nabla^\alpha p'_1 + \Delta' \boldsymbol{u}'_1,$$

$$\mathrm{D}^\alpha_t = \partial_{t^\alpha} + \overline{u}_0(s, N)\partial_{s^\alpha}, \quad \Delta' = \nabla' \cdot \nabla'. \tag{50}$$

From Eq. (48) it follows that p'_1 is independent of $\boldsymbol{x}'$ since $\boldsymbol{N}'_0$ vanishes, according to Eq. (45). By taking the curl with respect to $\boldsymbol{x}'$ one then obtains from Eq. (50)

$$\mathrm{D}'_t \boldsymbol{\omega}'_I = -\nabla' \times \boldsymbol{N}'_{I-1} - \mathrm{D}^\alpha_t \boldsymbol{\omega}'_1 - (\partial_N \overline{u}_0)(\boldsymbol{e}_n \partial_{z'} - \boldsymbol{e}_z \partial_{n'}) v'_1 + \Delta' \boldsymbol{\omega}'_1. \tag{51}$$

The right-hand sides of both Eqs. (51) and (49) do not explicitly depend on t'. With the same arguments leading to Eq. (48), then the Helmholtz decomposition of $\boldsymbol{\omega}'_I$ suggests that $\boldsymbol{u}'_I$ and, as a consequence of Eq. (50), p'_I exhibit no explicit t'-dependence too. In turn, the right-hand side of Eq. (51) must vanish. Therefore, Eq. (51) not only determines the quantity $\boldsymbol{u}'_{I-1}$, but can also be interpreted as a linear transport equation for the leading-order contribution $\boldsymbol{\omega}'_1$ to the vorticity with respect to the newly introduced time t^α and $\boldsymbol{x}'$. However, the motion which is affected by the viscous term in Eq. (3) is presumably also governed by convective terms which are nonlinear in the leading-order contribution $\boldsymbol{u}'_1$ to the velocity fluctuations. But, in view of Eq. (50), this is only possible by introducing a set of "intermediate micro-variables" $(\hat{t}, \hat{\boldsymbol{x}}) = (t, \boldsymbol{x})/\alpha^{3/2}$. Thus, the associated new length scale of $\mathcal{O}(\alpha^{3/2})$ is much larger than the viscosity-affected one, λ, but still smaller than the shear layer thickness of $\mathcal{O}(\alpha)$. We close the analysis by noting that this new length scale serves as a measure for the size of the large eddies in the wake region, and, in turn, of the mixing length. This fully agrees with the scaling of the latter found from the time-mean analysis, cf. Scheichl and Kluwick (2007b).

4. Conclusions and further outlook

We have demonstrated that turbulent bluff-body separation requires a streamwise velocity defect of $\mathcal{O}(1)$ as $\mathrm{Re} \to \infty$ in the fully turbulent main region of the oncoming boundary layer, as the classical assumption of a small velocity deficit is intrinsically tied to the idea of a firmly attached external potential flow, and, in turn, leads to a serious inconsistency in the asymptotic hierarchy of the flow, which originates from an asymptotically small vicinity of the rear stagnation point. On the other hand, for the large-defect boundary layer the limiting inviscid solution must be sought in the class of flows exhibiting a free streamline which departs smoothly from the surface. As one remarkable result strikingly contrasting a well-known finding in the theory of laminar separation, here the Brillouin–Villat condition is not met at the separation point. The formulation of the locally strong rotational/irrotational interaction of the separating flow with the external bulk flow is a topic of the current research. Future research activities include, amongst others, the asymptotic investigation of the unsteady motion, where particular emphasis should be placed on the rationally based modelling of turbulent boundary layers undergoing separation. Most important, as a first step in this direction, it has been shown here how the underlying boundary layer concept is strongly supported by such an analysis. Here we note that, to the authors' knowledge, the only attempt currently available in literature to treat turbulent boundary layers by tackling the full Navier–Stokes equations in the limit (1) from a rigorous asymptotic point of view must be attributed to Deriat and Guiraud (1986). As one physically appealing result of the present study, an inner length of $\mathcal{O}(\alpha^{3/2})$ reflecting the size of the large-scale eddies in the wake flow regime is found, which interestingly equals that of the mixing length, given by Scheichl and Kluwick (2007b). Although the analysis also predicts even larger eddies having a diameter of $\mathcal{O}(\alpha)$, those comparable to the mixing length scale determine the distance at which the flow starts to feel the presence of a confining wall. Remarkably, this interpretation of the mixing length not only fully agrees with the mathematical need to introduce a so-called inner wake layer having a thickness of $\mathcal{O}(\alpha^{3/2})$, see Scheichl and Kluwick (2007b). Moreover, it also conforms to the mixing length hypothesis (originally developed by Prandtl), where mid-size eddies are responsible for the turbulence transport from one fluid layer to the adjacent one, as proposed by Hinze (1975).

Finally, we stress that, notwithstanding the strongly encouraging progress made so far, the asymptotic analysis of the unsteady flow is still in a rather early stage. Therefore, its implications on turbulence modelling (of both attached and separating flow) can hardly be reliably projected for the time being. Also, comparison of the theory with both experimental and numerical data obtained for bluff-body flows—see e.g. the recent database provided by Braza et al. (2006)—is clearly a required task of future research efforts.

Acknowledgements

This research was granted by the Austrian Science Fund (Project no. P16555-N12), which is gratefully acknowledged. Also, the authors would like to thank Professor F.T. Smith (Department of Mathematics, University College London, London, UK) for stimulating discussions and the referees for their interesting and helpful comments.

References

Afzal, N., 1996. Wake layer in a turbulent boundary layer with pressure gradient: a new approach. In: Gersten, K. (Ed.), Asymptotic Methods for Turbulent Shear Flows at High Reynolds Numbers. Proceedings of the IUTAM Symposium, Bochum, Germany, June 28–30, 1995. Fluid Mechanics and its Applications, vol. 37. Kluwer Academic Publishers, Dordrecht, Boston, London, pp. 119–132.

Birkhoff, G., Zarantonello, E.H., 1957. Jets, Wakes, and Cavities. Applied Mathematics and Mechanics, vol. 2. Academic Press, New York.

Braza, M., Perrin, R., Hoarau, Y., 2006. Turbulence properties in the cylinder wake at high Reynolds numbers. Journal of Fluids and Structures 22 (6/7), 757–771.

Degani, A.T., 1996. Recent advances in wall-bounded shear flows. Survey of contributions from the USA (invited lecture). In: Gersten, K. (Ed.), Asymptotic Methods for Turbulent Shear Flows at High Reynolds Numbers. Proceedings of the IUTAM Symposium, Bochum, Germany, June 28–30, 1995. Fluid Mechanics and its Applications, vol. 37. Kluwer Academic Publishers, Dordrecht, Boston, London, pp. 119–132.

Deriat, E., Guiraud, J.-P., 1986. On the asymptotic description of turbulent boundary layers. Journal of Theoretical and Applied Mechanics 109–140 (Special issue on Asymptotic Modelling of Fluid Flows, English translation of: Journal de Mécanique Théorique et Appliquée).

Gurevich, M.I., 1966. The Theory of Jets in an Ideal Fluid. International Series of Monographs in Pure and Applied Mathematics, vol. 93. Pergamon Press, Oxford (original Russian book: 1961, Fizmatlit, Moscow).

Hinze, J.O., 1975. Turbulence, second ed. McGraw-Hill, New York.

Imai, I., 1953. Discontinuous potential flow as the limiting form of the viscous flow for vanishing viscosity. Journal of the Physical Society of Japan 8 (3), 399–402.

Mellor, G.L., 1972. The large Reynolds number, asymptotic theory of turbulent boundary layers. International Journal of Engineering Science 10 (10), 851–873.

Melnik, R.E., 1989. An asymptotic theory of turbulent separation. Computers and Fluids 17 (1), 165–184.

Neish, A., Smith, F.T., 1992. On turbulent separation in the flow past a bluff body. Journal of Fluid Mechanics 241, 443–467.

Scheichl, B., Kluwick, A., 2007a. On turbulent marginal boundary layer separation: how the half-power law supersedes the logarithmic law of the wall. International Journal of Computing Science and Mathematics 1 (2/3/4), 343–359 (Special Issue on Problems Exhibiting Boundary and Interior Layers).

Scheichl, B., Kluwick, A., 2007b. Turbulent marginal separation and the turbulent Goldstein problem. AIAA Journal 45 (1), 20–36 (see also: AIAA Technical Paper 2005-4936).

Scheichl, B.F., 2001. Asymptotic theory of marginal turbulent separation. Ph.D. Thesis, Vienna University of Technology, Vienna, Austria.

Smith, F.T., 1977. The laminar separation of an incompressible fluid streaming past a smooth surface. Proceedings of the Royal Society of London, Series A 356 (1687), 443–463.

Sychev, V.V., 1972. Laminar separation. Fluid Dynamics 7 (3), 407–417 (English translation of: Izvestiya Akademii Nauk SSSR—Mekhanika Zhidkosti i Gaza 3, 1972, 47–59).
Sychev, Vik.V., 1983. Asymptotic theory of turbulent separation. Fluid Dynamics 18 (4), 532–538 (English translation of: Izvestiya Akademii Nauk SSSR—Mekhanika Zhidkosti i Gaza 18 (4), 1983, 47–54).
Sychev, Vik.V., 1987. Theory of self-induced separation of a turbulent boundary layer. Fluid Dynamics 22 (3), 371–379 (English translation of: Izvestiya Akademii Nauk SSSR—Mekhanika Zhidkosti i Gaza 22 (3), 1987, 51–60).
Tsahalis, D.T., Telionis, D.P., 1975. On the behavior of turbulent boundary layers near separation. AIAA Journal 13 (10), 1261–1262 (Synoptics).